

Complete Group Classification of Systems of Two Nonlinear Second-Order Ordinary Differential Equations of the Form $\mathbf{y}'' = \mathbf{F}(\mathbf{y})$

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Abstract

Extensive work has been done on the group classification of systems of equations in the literature. This paper identifies the gap in the literature which concerns the group classification of systems of two nonlinear second-order ordinary differential equations. We provide a complete group classification of systems of two ordinary differential equations of the form, $\mathbf{y}'' = \mathbf{F}(\mathbf{y})$, which occur in many physical applications using two approaches which form the essence of this paper.

Key words: Group classification, autonomous systems, nonlinear systems, admitted Lie group

1 Introduction

Systems of autonomous nonlinear second-order ordinary differential equations, where the independent variable, usually assumed to be time, does not appear on the right hand side of the system, arise in various physical problems. This effectively assumes that the laws of nature which hold true in the present are presumably applicable in the past and the future. Hence,

like all other systems of differential equations, the study of their symmetry structure poses an important role as their presence in a system allows one to reduce the order of the studied equations and also to find general solutions in quadratures.

Group classification studies, dating more than a century back, were first initiated by the founder of symmetry analysis, Sophus Lie [1,2,3,4]. These studies were long forgotten until Ovsiannikov [5,6] revived the work five decades ago. Lie's works put emphasis on tackling the group classification in two ways: 1) the direct way and 2) the indirect way also known as the algebraic approach. The direct way involves directly finding solutions of the determining equations and allows one to study all possible admitted Lie algebras without omission. On the other hand the indirect way involves solving the determining equations up to finding relations between constants defining admitted generators. The algebraic approach, as in the studies [7,8,9,10]^{*}, takes into account the algebraic properties of an admitted Lie group and the knowledge of the algebraic structure of the admitted Lie algebras in order to allow significant simplification of the group classification. In one of Lie's works [1], he gave a complete group classification of a single second-order ordinary differential equation of the form $y'' = f(x, y)$. Later on Ovsiannikov [11] did this group classification in a different way. The method he used, now also known as the direct approach, involved a two-step technique where the determining equations were first simplified through exploiting equivalence transformations and later on solved for the reduced cases of the generators. The same technique was used in a study conducted in [12] to classify a more general case of equations of the form $y'' = P_3(x, y; y')$, where $P_3(x, y; y')$ is a polynomial of a third degree with respect to the first-order derivative y' . Observe that sometimes difficulties arise in using the direct approach. Sometimes it is difficult to select or tease out equivalent cases with respect to equivalence transformations. As is observed in the classification of a general scalar second-order ordinary differential equation of the form $y'' = f(x, y; y')$, the application of the direct technique gives rise to overwhelming difficulties. In this study both the direct and indirect techniques are employed.

Apart from dealing with classification problems there is a significant amount

^{*} See also references therein.

of research that deals with the dimension and structure of symmetry algebras of linearizable ordinary differential equations [7,13,14,15,16,17]. This is also of importance since some nonlinear equations appear in disguised forms.

In addition to extensive studies on properties of scalar second-order ordinary differential equations, there are also several researchers committed to studying systems of two linear second-order ordinary differential equations [18,19,17,20,21,22,23,24,25]. Surprisingly, the group classification of systems of two nonlinear second-order ordinary differential equations has not yet been exhausted, in particular, the group classification of systems of two autonomous nonlinear second-order ordinary differential equations is not yet complete. Hence this paper considers the group classification of systems of two autonomous nonlinear second-order ordinary differential equations of the form

$$\mathbf{y}'' = \mathbf{F}(\mathbf{y}). \quad (1)$$

The system studied here is a generalization of Lie's study [2]. Studied cases such as systems of two linear second-order ordinary differential equations and the degenerate case which is equivalent to the following

$$y'' = F(x, y, z), \quad z'' = 0 \quad (2)$$

are omitted from this paper. We call systems that are equivalent to these cases as reducible systems and irreducible otherwise.

The paper is organized as follows. A preliminary study of systems of two nonlinear second-order ordinary differential equations is tackled first and is followed by the subsequent group classification applied to autonomous systems (1) of two second-order ordinary differential equations. The group classification is divided into two parts depending on the coefficient of the infinitesimal generator. The direct approach is applied on one case while a combination of the optimal system of subalgebras and direct approach is applied to the other case. The latter part of the paper lists the different cases with their respective results and is then followed by the conclusion.

2 Background study of systems of the form $\mathbf{y}'' = \mathbf{F}(x, \mathbf{y})$

This section focuses on systems of two nonlinear second-order ordinary differential equations of the form [22,25]

$$\mathbf{y}'' = \mathbf{F}(x, \mathbf{y}), \quad (3)$$

where

$$\mathbf{y} = \begin{pmatrix} y \\ z \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} F(x, y, z) \\ G(x, y, z) \end{pmatrix}.$$

2.1 Equivalence transformations

System (3) has the following equivalence transformations:

- (1) a linear change of the dependent variables $\tilde{\mathbf{y}} = P\mathbf{y}$ with constant non-singular 2×2 matrix P ;
- (2) the change $\tilde{y} = y + \phi(x)$ and $\tilde{z} = z + \psi(x)$; and
- (3) the transformation related with the change $\tilde{x} = \phi(x)$, $\tilde{y} = y\psi(x)$, $\tilde{z} = z\psi(x)$, where the functions $\phi(x)$ and $\psi(x)$ satisfy the condition $\frac{\phi''}{\phi'} = 2\frac{\psi'}{\psi}$.

2.2 Determining equations

The determining equations in matrix form for irreducible systems of the form (3) are given by

$$2\xi\mathbf{F}_x + 3\xi'\mathbf{F} + (((A + \xi'E)\mathbf{y} + \zeta) \cdot \nabla)\mathbf{F} - A\mathbf{F} = \xi'''\mathbf{y} + \zeta'', \quad (4)$$

where the matrix $A = (a_{ij})$ is constant. The associated infinitesimal generator is

$$X = 2\xi(x)\partial_x + (A\mathbf{y} + \zeta(x)) \cdot \nabla,$$

where $\nabla = (\partial_y, \partial_z)^t$ and " \cdot " means the scalar product $\mathbf{b} \cdot \nabla = b_i \partial_{y_i}$, where the summation with respect to the repeated index is used [22].

The equivalence transformation (1) with linear change $\tilde{\mathbf{y}} = P\mathbf{y}$, when applied to equations (3), reduces equations (4) and its associated infinitesimal generator to the same form with the matrix A and the vector ζ changed. Equations (3) become

$$\tilde{\mathbf{y}}'' = \tilde{\mathbf{F}}(x, \tilde{\mathbf{y}})$$

with

$$\tilde{\mathbf{F}}(x, \tilde{\mathbf{y}}) = P\mathbf{F}(x, P\tilde{\mathbf{y}}),$$

and the partial derivatives with respect to the variables \mathbf{y} are also changed as follows

$$\mathbf{b} \cdot \nabla = (P\mathbf{b}) \cdot \tilde{\nabla}.$$

Consequently, the determining equations (4) become

$$2\xi\tilde{\mathbf{F}}_x + 3\xi'\tilde{\mathbf{F}} + (((\tilde{A} + \xi'E)\tilde{\mathbf{y}} + \tilde{\zeta}) \cdot \tilde{\nabla})\tilde{\mathbf{F}} - \tilde{A}\tilde{\mathbf{F}} - \xi'''\tilde{\mathbf{y}} - \tilde{\zeta}'' = 0,$$

where

$$\tilde{A} = PAP^{-1}, \quad \tilde{\zeta} = P\zeta$$

and the associated infinitesimal generator is also changed as follows

$$X = 2\xi(x)\partial_x + (\tilde{A}\tilde{\mathbf{y}} + \tilde{\zeta}(x)) \cdot \tilde{\nabla}.$$

As in [22,25] and in the succeeding pages, this transformation places a very important role in the group classification process.

From the study [22], the systems of two nonlinear second-order ordinary differential equations are equivalent to one of the following 10 types listed below. Looking closely at these systems, there is a necessity to conduct an initial study where the systems of two equations do not depend on x . This forms the core of this paper.

F and G

1. $F = e^{ax} f(u, v),$
 $G = e^{bx} g(u, v)$
2. $F = e^{ax} (\cos(cx) f(u, v) + \sin(cx) g(u, v)),$
 $G = e^{ax} (-\sin(cx) f(u, v) + \cos(cx) g(u, v))$
3. $F = e^{ax} (f(u, v) + xg(u, v)),$
 $G = e^{ax} g(u, v)$
4. $F = (y + h_1(x)) f(x, v) - h_1'',$
 $G = (z + h_2(x)) g(x, v) - h_2''$
5. $F = (y + h_1(x)) f(x, v) - h_1''(x),$
 $G = h_2''(x) \ln(y + h_1(x)) + g(x, v)$
6. $F = \frac{h_1''(x)}{h_1(x)} y + f(x, v),$
 $G = \frac{h_2''(x)}{h_1(x)} y + g(x, v)$
7. $F = e^{au} (\cos(cu) f(x, v) + \sin(cu) g(x, v)),$
 $G = e^{au} (-\sin(cu) f(x, v) + \cos(cu) g(x, v))$
8. $F = \frac{y}{z + h_1(x)} f(x, v) + g(x, v),$
 $G = -h_1''(x) + f(x, v)$
9. $F = \frac{h_2''(x)}{2} u^2 + u f(x, v) + g(x, v),$
 $G = -h_1''(x) + h_2''(x) u + f(x, v)$
10. $F = e^u (u f(x, v) + g(x, v)),$
 $G = e^u f(x, v)$

Relations and conditions

- $$u = ye^{-ax}, \quad v = ze^{-bx},$$
- a, b are constant
- $$u = e^{-ax} (y \cos(cx) - z \sin(cx)),$$
- $$v = e^{-ax} (y \sin(cx) + z \cos(cx)),$$
- $a, c \neq 0$ are constant
- $$u = e^{-ax} (y - zx), \quad v = ze^{-ax},$$
- a is constant
- $$v = (z + h_2(x))(y + h_1(x))^\alpha,$$
- $\alpha \neq 0$ is constant
- $$v = z - h_2(x) \ln(y + h_1(x))$$
- $$v = z - \frac{h_2(x)}{h_1(x)} y, \quad h_1(x) \neq 0$$
- $$y = ve^{au} \sin(cu), \quad z = e^{au} \cos(cu),$$
- $a, c \neq 0$ are constant
- $$v = z + h_1(x)$$
- $$u = \frac{z + h_1(x)}{h_2(x)}, \quad v = y - \frac{(z + h_1(x))^2}{h_2(x)},$$
- $h_2(x) \neq 0$
- $$y = uve^u, \quad z = ve^u$$

For all the cases, h_1, h_2, f and g are arbitrary functions of their arguments.

3 Autonomous systems (1) of two nonlinear second-order ordinary differential equations and their group classification

Since for autonomous systems $\mathbf{F}_x = 0$, the determining equations for autonomous systems have the form

$$3\xi'\mathbf{F} + (((A + \xi'E)\mathbf{y} + \zeta) \cdot \nabla)\mathbf{F} - A\mathbf{F} - \xi'''\mathbf{y} - \zeta'' = 0. \quad (5)$$

This also implies that the generator ∂_x is admitted by system (1).

Differentiating the determining equations (5) with respect to x , the group classification study is reduced into two cases, namely,

- (1) the case with at least one admitted generator with $\xi'' \neq 0$; and
- (2) the case where all admitted generators have $\xi'' = 0$.

For the first case, the direct approach by Lie is utilized, whereas for the second case, a combination of the optimal system of subalgebras of the Lie algebra and the direct method is used.

3.1 Systems admitting at least one admitted generator with $\xi'' \neq 0$

For the case with at least one generator, $\xi'' \neq 0$, we initially consider the differentiated determining equations (5) with respect to x and divide them by ξ'' . The determining equations become

$$3\mathbf{F} + \left(\left(\mathbf{y} + \frac{\zeta'}{\xi''} \right) \cdot \nabla \right) \mathbf{F} - \frac{\xi^{(4)}}{\xi''}\mathbf{y} - \frac{\zeta'''}{\xi''} = 0. \quad (6)$$

Fixing x , and shifting y and z , equations (6) are reduced to

$$3\mathbf{F} + (\mathbf{y} \cdot \nabla)\mathbf{F} - a\mathbf{y} - \mathbf{b} = 0$$

where vector $\mathbf{b} = (b, c)^t$, with a, b, c constant.

The general solution of these equations is

$$\begin{aligned} F &= \frac{b}{3} + \frac{ay}{4} + y^{-3}f(u), \\ G &= \frac{c}{3} + \frac{az}{4} + z^{-3}g(u), \end{aligned} \quad (7)$$

where $u = z/y$ and $f'g' \neq 0$. It is easy to see that if $f'g' = 0$, the studied system is equivalent to a reducible case. The functions F and G are then substituted into the determining equations (5). The determining equations are then solved directly in order to find generators admitted by equations (1). The first part of the determining equations is given as follows:

$$\xi''' - a\xi' = 0, \quad (8)$$

$$\begin{aligned} (\zeta_1 u - \zeta_2)f_u + 3\zeta_1 f &= 0, \\ (u^2\zeta_1 - u\zeta_2)g_u + 3\zeta_2 g &= 0, \end{aligned} \quad (9)$$

$$\begin{aligned} 12\zeta_1'' - 12b\xi' - 3a\zeta_1 + 4a_{11}b + 4a_{12}c &= 0, \\ 12\zeta_2'' - 12c\xi' - 3a\zeta_2 + 4a_{21}b + 4a_{22}c &= 0, \end{aligned} \quad (10)$$

$$\begin{aligned} (a_{11}u^4 + a_{12}u^5 - a_{21}u^3 - a_{22}u^4)f_u + (4a_{11}u^3 + 3a_{12}u^4)f + a_{12}g &= 0, \\ (a_{11}u^2 + a_{12}u^3 - a_{21}u - a_{22}u^2)g_u + a_{21}u^4f + (3a_{21} + 4a_{22}u)g &= 0. \end{aligned} \quad (11)$$

From equation (8), it can be seen that the general solution of ξ depends on three values of a , i.e., $a = 0$, $a = -p^2$ and $a = p^2$, where $p \neq 0$. For $a = 0$, the general solution of ξ is

$$\xi = \xi_2 x^2 + \xi_1 x + \xi_0,$$

where $\xi_2 \neq 0$, ξ_1 , ξ_0 are constant. For $a = -p^2$, the general solution of ξ is

$$\xi = \xi_1 \cos(px) + \xi_2 \sin(px) + \xi_0,$$

where $\xi_2 \neq 0$, $\xi_1 \neq 0$, ξ_0 are constant. Lastly, for $a = p^2$, the general solution of ξ is

$$\xi = \xi_1 e^{-px} + \xi_2 e^{px} + \xi_0,$$

where $\xi_2 \neq 0$, $\xi_1 \neq 0$, ξ_0 are constant. Subsequently the determining equations (9) lead to the study of two cases where: (1) there exists a generator with $\zeta_1 \neq 0$ and (2) all generators have $\zeta_1 = 0$.

Considering the case where there exists a generator with $\zeta_1 \neq 0$, we divide by ζ_1 and differentiate the equations (9) with respect to x to obtain, $\zeta_2 = k\zeta_1$, where k is a constant. Substituting this back to equations (9), one obtains $f = f_0(u - k)^{-3}$ and $g = g_0u^3(u - k)^{-3}$. Also, differentiating equations (10) with respect to x , it follows that $c = kb$. From here, one can verify that this is a reducible case.

Consider that all generators have $\zeta_1 = 0$. From equations (9), it follows that $\zeta_2 = 0$. Differentiating equations (10) with respect to x , it immediately follows that $b = c = 0$. From equations (11), the equivalence transformation $\tilde{y} = Py$, where P is a constant nonsingular 2×2 matrix, is utilized in order to obtain the general solution of f and g . Note that the constant matrix A is reduced to one of the following real-valued Jordan forms

$$J_1 = \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix}, J_2 = \begin{pmatrix} a_{11} & 1 \\ -1 & a_{11} \end{pmatrix}, J_3 = \begin{pmatrix} a_{11} & 1 \\ 0 & a_{11} \end{pmatrix}. \quad (12)$$

The general solutions for f and g are listed as follows:

Jordan form	f	g
J_1	$f_0u^{-(4+\frac{4}{\gamma-1})}$	$f_1u^{-\frac{4}{\gamma-1}}$
J_2	$(f_0y - f_1z)\tau(y, z)$	$(f_0z + f_1y)\tau(y, z)$
J_3	$e^{-u}(f_0u^{-4} + f_1u^{-3})$	f_0e^{-u} .

In this instance, $\tau(y, z) = e^{4\alpha \arctan \frac{z}{y}}(y^2 + z^2)^{-2}$, and $f_0 \neq 0$, $f_1 \neq 0$, $\gamma \neq 1$, $\alpha \neq 1$ are constant.

Excluding reducible systems, the classes of functions F and G of equations (5) admitting a corresponding Lie group and the extension of the kernel of the admitted Lie algebras are obtained as seen in Table 1. The kernel of the admitted Lie algebras consists of the generator $X_1 = \partial_x$, which will be

omitted on the list. The extension of the kernel is listed as follows:

$$\begin{aligned}
Y_2 &= 2x\partial_x + y\partial_y + z\partial_z, & Y_3 &= x(x\partial_x + y\partial_y + z\partial_z), \\
Y_4 &= \gamma y\partial_y + z\partial_z, & Y_5 &= (\alpha y - z)\partial_y + (y + \alpha z)\partial_z, \\
Y_6 &= (y + 4z)\partial_y + z\partial_z, & Y_7 &= \cos 2x\partial_x - \sin 2x(y\partial_y + z\partial_z), \\
Y_8 &= \sin 2x\partial_x + \cos 2x(y\partial_y + z\partial_z), & Y_9 &= e^{-2x}(\partial_x - (y\partial_y + z\partial_z)), \\
Y_{10} &= e^{2x}(\partial_x + y\partial_y + z\partial_z).
\end{aligned}$$

The Lie algebras Y_2, Y_3, Y_7, Y_8, Y_9 , and Y_{10} are associated with the coefficient ξ and the Lie algebras Y_4, Y_5 , and Y_6 are related to the type of Jordan form of matrix A .

3.2 Systems where all admitted generators have $\xi'' = 0$

Note that the action of equivalence transformations coincides with the action of group automorphisms. For the direct approach, sometimes it is difficult to select out equivalent cases with respect to equivalence transformations. Fortunately, if the algebraic structure of the admitted Lie algebra is known, then using the algebraic approach aids in simplifying the group classification problem. Thus, for finding the group classification of systems of two autonomous nonlinear second-order ordinary differential equations with all admitted generators satisfying $\xi'' = 0$, the one-dimensional optimal system of one parameter subgroups is utilized and is proceeded by the direct approach. Firstly, the two-step algorithm of Ovsianikov [26] is employed here, for which the optimal systems of subgroup and group invariant solutions are reconstructed. Invariant solutions are then substituted back into the determining equations (5) where the direct method is used to find all possible admitted Lie algebras.

Firstly returning to the analysis of the determining equations (5), since $\xi'' = 0$, it follows that $\xi = k_1 + k_2x$, where k_1 and k_2 are constant. This property of the coefficient forces ζ to become constant.^{★★} The determining

^{★★}See in Appendix.

equations (5) are then reduced to

$$3k_2\mathbf{F} + (((A + k_2E)\mathbf{y} + \mathbf{k}) \cdot \nabla)\mathbf{F} - A\mathbf{F} = 0 \quad (13)$$

with the following admitted generator

$$X = 2(k_1 + k_2x)\partial_x + (A\mathbf{y} + \mathbf{k}) \cdot \nabla \quad (14)$$

where the matrix A is a vector \mathbf{k} , k_1 and k_2 are constant. By rewriting (14), the generator can be represented as

$$X = \sum_{i=1}^8 c_i X_i \quad (15)$$

where $c_i (i = 1, \dots, 8)$ are constant. Corresponding to the constants $c_i (i = 1, \dots, 8)$, the basis operators of the Lie algebra are as follows:

$$\begin{aligned} X_1 &= \partial_x & X_2 &= x\partial_x & X_3 &= \partial_y & X_4 &= \partial_z \\ X_5 &= y\partial_y & X_6 &= z\partial_z & X_7 &= z\partial_y & X_8 &= y\partial_z. \end{aligned} \quad (16)$$

From here, the one-dimensional optimal system of one parameter subgroups of the main group of system (1) with $\xi'' = 0$ is constructed. The commutators of the basis operators are

$$\begin{aligned} [X_1, X_2] &= X_1, & [X_5, X_7] &= -X_7, \\ [X_3, X_5] &= X_3, & [X_5, X_8] &= X_8, \\ [X_3, X_8] &= X_4, & [X_6, X_7] &= X_7, \\ [X_4, X_6] &= X_4, & [X_6, X_8] &= -X_8, \\ [X_4, X_7] &= X_3, & [X_7, X_8] &= X_6 - X_5. \end{aligned} \quad (17)$$

The following inner automorphisms $A_i (i = 1, \dots, 8)$ of the above Lie algebra

are found without difficulties:

$$\begin{aligned}
A_1 : \hat{c}_1 &= c_1 - a_1 c_2, \\
A_2 : \hat{c}_1 &= e^{a_2} c_1, \\
A_3 : \hat{c}_3 &= c_3 - a_3 c_5, \quad \hat{c}_4 = c_4 - a_3 c_8, \\
A_4 : \hat{c}_3 &= c_3 - a_4 c_7, \quad \hat{c}_4 = c_4 - a_4 c_6, \\
A_5 : \hat{c}_3 &= e^{a_5} c_3, \quad \hat{c}_7 = e^{a_5} c_7 \quad \hat{c}_8 = e^{-a_5} c_8, \\
A_6 : \hat{c}_4 &= e^{a_6} c_4, \quad \hat{c}_7 = e^{-a_6} c_7 \quad \hat{c}_8 = e^{a_6} c_8, \\
A_7 : \hat{c}_3 &= c_3 + a_7 c_4, \quad \hat{c}_5 = c_5 + a_7 c_8, \quad \hat{c}_6 = c_6 - a_7 c_8, \\
&\hat{c}_7 = c_7 - a_7^2 c_8 + a_7 c_6 - a_7 c_5, \\
A_8 : \hat{c}_4 &= c_4 + a_8 c_3, \quad \hat{c}_5 = c_5 - a_8 c_7, \quad \hat{c}_6 = c_6 + a_8 c_7, \\
&\hat{c}_8 = c_8 - a_8^2 c_7 - a_8 c_6 + a_8 c_5.
\end{aligned} \tag{18}$$

Note that a_i ($i = 1, \dots, 8$) are the parameters on which the transformations of the group depend on. Apart from these automorphisms, we have the following involutions:

$$\begin{aligned}
E_1 : \bar{z} &= -z \mid \quad \bar{c}_4 = -c_4, \bar{c}_7 = -c_7, \bar{c}_8 = -c_8; \\
E_2 : \bar{y} &= -y \mid \quad \bar{c}_3 = -c_3, \bar{c}_7 = -c_7, \bar{c}_8 = -c_8; \\
E_3 : \bar{x} &= -x \mid \quad \bar{c}_1 = -c_1; \\
E_4 : \bar{y} &= z, \bar{z} = y \mid \quad \bar{c}_3 = c_4, \bar{c}_4 = c_3, \bar{c}_5 = c_6, \bar{c}_6 = c_5, \bar{c}_7 = c_8, \bar{c}_8 = c_7.
\end{aligned}$$

We study the way in which the coefficients of equation (15) are changed under the action of inner automorphisms of the group above. Here and further on, only changeable coordinates of the generator are presented. Looking closely at the commutators, the Lie algebra L_8 , which is composed of the generators X_i ($i = 1, \dots, 8$), can be split into 2 subalgebras $L_2 \oplus L_6 = \{X_1, X_2\} \oplus \{X_3, X_4, X_5, X_6, X_7, X_8\}$. Note also that L_6 can be decomposed further to $L_4 \oplus I_2 = \{X_5, X_6, X_7, X_8\} \oplus \{X_3, X_4\}$, where L_4 makes up a 4-dimensional subalgebra and I_2 is ideal.

Let us first study the 4-dimensional subalgebra $L_4 = \{X_5, X_6, X_7, X_8\}$. We

consider this study here due to a misprint found in the classification of this Lie algebra in [27]. Now consider the operator X of a one parameter subgroup of the form

$$X = c_5X_5 + c_6X_6 + c_7X_7 + c_8X_8. \quad (19)$$

Automorphisms A_5 up to A_8 are made use of in order to find the one-dimensional optimal system of subalgebras of this Lie algebra. From the automorphisms A_5 and A_6 , one can find the invariant $\bar{c}_7\bar{c}_8 = c_7c_8$, which leads one to consider the following cases:

$$(a) \ c_7c_8 > 0$$

$$(b) \ c_7c_8 < 0$$

$$(c) \ c_7c_8 = 0.$$

Then utilizing the invariant of A_7 and A_8 , which is $\bar{c}_5 + \bar{c}_6 = c_5 + c_6$, one can obtain relations of c_5 and c_6 . Upon further computations using automorphisms, it can be verified that for case (a), the coefficients of equation (19) satisfy $c_5 - c_6 \neq 0$, $c_7 = 0$ and $c_8 = 0$. For case (b), it follows that $c_5 = c_6$, $c_7 = -1$ and $c_8 = 1$. For case (c) if $c_5 \neq c_6$ then $c_7 = 0$ and $c_8 = 0$. If $c_5 = c_6$, then $c_7 = 1$ and $c_8 = 0$. The involutions are also utilized. Hence, the following one-dimensional optimal system of subalgebras of the Lie algebra L_4 is obtained:

1. $X_5 + \alpha X_6$ where $-1 \leq \alpha \leq 1$
 2. $\alpha(X_5 + X_6) + X_8 - X_7$ where $\alpha \geq 0$
 3. $\beta(X_5 + X_6) + X_7$ where $\beta = 0, 1$
 4. 0.
- (20)

Note that the 0 element is considered on this list [26]. There is a necessity to include this element on the list as when the direct sum $L_4 \oplus I_2$ is applied, more subalgebras of the Lie algebra L_6 may appear on the list.

Remark: As the action of the above automorphisms coincides with the action of the equivalence transformations, it is possible to get the optimal system of one-dimensional subalgebras of the Lie algebra L_4 using

the latter. From the determining equations (13) and the utilization of the equivalence transformation $\tilde{y} = Py$, where P is a nonsingular 2×2 matrix with constant entries, the matrix of coefficients of (19)

$$\begin{pmatrix} c_5 & c_7 \\ c_8 & c_6 \end{pmatrix}$$

is reduced to one of its real-valued Jordan forms (12). Looking closely at (20), subalgebra 1. coincides with Jordan matrix J_1 , subalgebra 2. coincides with Jordan matrix J_2 , and subalgebra 3. coincides with Jordan matrix J_3 .

3.2.1 Optimal system of subalgebras of the algebra $L_6 = \{X_3, X_4, X_5, X_6, X_7, X_8\}$

After obtaining the one-dimensional optimal system (20) of subalgebras of the Lie algebra $L_4 = \{X_5, X_6, X_7, X_8\}$, the next step is to combine L_4 with the ideal $I_2 = \{X_3, X_4\}$. Here, again Ovsiannikov's two-step method [26] is applied. Hence, for the study of the one-dimensional subalgebras of the Lie algebra L_6 , the study is reduced to analyzing the following elements:

1. $c_3X_3 + c_4X_4 + X_5 + \alpha X_6$ where $-1 \leq \alpha \leq 1$
 2. $c_3X_3 + c_4X_4 + \alpha(X_5 + X_6) + X_8 - X_7$ where $\alpha \geq 0$
 3. $c_3X_3 + c_4X_4 + \beta(X_5 + X_6) + X_7$ where $\beta = 0, 1$
 4. $c_3X_3 + c_4X_4$.
- (21)

Using automorphisms A_3 and A_4 and the involutions, the list of one-dimensional subalgebras of the Lie algebra $L_6 = \{X_3, X_4, X_5, X_6, X_7, X_8\}$

is obtained as follows:

1. $X_5 + \alpha X_6$ where $-1 \leq \alpha \leq 1$
 2. $X_4 + X_5$
 3. $X_8 - X_7$
 4. $\beta X_3 + \alpha(X_5 + X_6) + X_8 - X_7$ where $\beta = -1, 0, 1, \alpha > 0$
 5. $\beta X_4 + X_7$ where $\beta = 0, 1$
 6. $X_5 + X_6 + X_7$
 7. X_3
 8. 0.
- (22)

Again, it is necessary to study the element 0 of the subalgebras of the Lie algebra L_6 as this may generate additional elements when L_6 is combined with L_2 .

3.2.2 Optimal system of subalgebras of the algebra $L_8 = \{X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8\}$

Combining L_6 with L_2 and keeping in mind that for autonomous systems X_1 is already admitted, the following elements comprise the list of one-dimensional subalgebras of the Lie algebra L_8 :

1. $\gamma X_2 + X_5 + \alpha X_6$ where $-1 \leq \alpha \leq 1$
 2. $\gamma X_2 + X_4 + X_5$
 3. $\gamma X_2 + X_8 - X_7$
 4. $\gamma X_2 + \beta X_3 + \alpha(X_5 + X_6) + X_8 - X_7$ where $\beta = -1, 0, 1, \alpha > 0$
 5. $\gamma X_2 + \beta X_4 + X_7$ where $\beta = 0, 1$
 6. $\gamma X_2 + X_5 + X_6 + X_7$
 7. $\gamma X_2 + X_3$
 8. X_2 .
- (23)

Using this list of subalgebras, the next step is to obtain invariant solutions F and G of the determining equations (13). These functions are substituted into the determining equations (5), which are solved completely in order to find all other generators admitting equations (1).

3.2.3 Representations of systems of two nonlinear second-order ordinary differential equations with all generators having $\xi'' = 0$

From (15), c_i ($i = 1, \dots, 8$) are the coefficients of the generator chosen from the above list of subalgebras. Only one subalgebra is presented in this paper as computations for the other subalgebras are done in a similar way.

3.2.3.1 Subalgebra 1. with the generator $\gamma X_2 + X_5 + \alpha X_6$ where $-1 \leq \alpha \leq 1$. For this case, the determining equations (13) become

$$yF_y + \alpha zF_z - (2\gamma - 1)F = 0$$

$$yG_y + \alpha zG_z - (2\gamma - \alpha)G = 0.$$

The general solution of these equations is

$$F(y, z) = f(u)y^{1-2\gamma} \text{ and } G(y, z) = g(u)y^{\alpha-2\gamma},$$

where $u = y^\alpha/z$. Notice that $f' \neq 0$ else it is equivalent to a reducible case. Substituting these functions to the determining equations (5), the following initial determining equations are obtained

$$\begin{aligned} & y^{2\alpha}a_{12}(\alpha uf' + (1 - 2\gamma)f - ug) \\ & + y^{\alpha+1}u((\alpha a_{11} + (\alpha - 1)\xi_1 - a_{22})uf' - 2(\gamma a_{11} + (\gamma - 2)\xi_1)f) \\ & + y^\alpha \zeta_1 u(\alpha uf' + (1 - 2\gamma)f) - y^2 a_{21} u^3 f' - y \zeta_2 u^3 f' = 0, \\ & y^{2\alpha}a_{12}(\alpha ug' + (\alpha - 2\gamma)g) \\ & + y^{\alpha+1}u((\alpha a_{11} + (\alpha - 1)\xi_1 - a_{22})ug' - ((\alpha - 2\gamma)a_{11} + (\alpha - 2\gamma + 3)\xi_1 - a_{22})g) \\ & + y^\alpha u \zeta_1 (\alpha ug' + (\alpha - 2\gamma)g) - y^2 a_{21} u(u^2 g' + f) - y \zeta_2 u^3 g' = 0 \end{aligned}$$

In order to split these determining equations, one needs to study relations between the powers of y . Thus, one needs to evaluate the following cases: (1) $\alpha = 0$, (2) $\alpha = \frac{1}{2}$, (3) $\alpha = 1$ and (4) $\alpha \neq 0, \frac{1}{2}, 1$.

- (1) Consider when $\alpha = 0$. After splitting with respect to y , it can be verified that $a_{21} = 0$ and one is left with the following determining equations

$$\begin{aligned} ((1 - 2\gamma)(\zeta_1 u + a_{12}))f - a_{12}ug &= 0, \\ 2(\gamma a_{11} + (\gamma - 2)\xi_1)f + (a_{22} + \xi_1 + \zeta_2 u)uf' &= 0, \\ \gamma(a_{12} + \zeta_1 u)g &= 0, \\ (2\gamma a_{11} + (2\gamma - 3)\xi_1 + a_{22})g + (a_{22} + \xi_1 + \zeta_2 u)ug' &= 0. \end{aligned} \tag{24}$$

From the third equation, notice that if $\gamma = 0$, G becomes a function solely of z and hence, this case is reducible. Thus, it follows that $a_{12} = 0$ and $\zeta_1 = 0$. Dividing the second equation by f' and u , and differentiating it with respect to u 2 times, one can study the following cases:

- (a) $\left(\frac{f}{uf'}\right)'' \neq 0$ and
(b) $\left(\frac{f}{uf'}\right)'' = 0$.

For the case when $\left(\frac{f}{uf'}\right)'' \neq 0$, it follows that $a_{11} = \xi_1 \frac{(2 - \gamma)}{\gamma}$. Consequently, $\zeta_2 = 0$ and $a_{22} = -\xi_1$. Substituting this into the determining equations (24), no other extensions of the generator are obtained apart from the studied subalgebra.

For the case when $\left(\frac{f}{uf'}\right)'' = 0$, it follows that $\frac{f}{uf'} = \kappa u + \beta$. Furthermore, the general solution of this depends on β . Thus, one needs to study whether $\beta \neq 0$ or $\beta = 0$.

- (i) For the case when $\beta \neq 0$, the general equation for f (with a possible shift) is $f_0 \left(\frac{1}{u}\right)^\beta$. Substituting this into the determining equations, one gets $a_{22} = \frac{(2\gamma - \beta - 4)\xi_1 + 2\gamma a_{11}}{\beta}$ and

$\zeta_2 = \frac{2\kappa(\gamma a_{11} + (\gamma - 2)\xi_1)}{\beta}$. Consequently, the general solution for g is $g_0 \left(\frac{1}{u}\right)^{\beta+1}$. The extension $\beta X_5 + 2\gamma X_6$ is obtained along with the studied subalgebra.

(ii) For the case when $\beta = 0$, it follows that $\kappa \neq 0$. Hence, the general equation for f is $f_0 e^{\kappa/u}$. Substituting this into the determining equations, one obtains $\zeta_2 = \frac{2(\gamma a_{11} + (\gamma - 2)\xi_1)}{\kappa}$ and $a_{22} = -\xi_1$. Consequently, the general solution for g is $g_0 e^{\kappa/u}$. The extension $\kappa X_5 + 2\gamma X_4$ is obtained aside the studied subalgebra.

(2) For the case when $\alpha = \frac{1}{2}$, after splitting with respect to y , it follows that $a_{21} = 0$. Also, since $(1 - 4\gamma)g + ug' = 0$ leads to a reducible case it then follows that $\zeta_1 = 0$. The remaining determining equations are

$$2a_{12}(1 - 2\gamma)f - 2a_{12}ug + (a_{12} - 2\zeta_2 u^2)uf' = 0,$$

$$4((2 - \gamma)\xi_1 - \gamma a_{11})f + (a_{11} - 2a_{22} - \xi_1)uf' = 0,$$

$$(1 - 4\gamma)a_{12}g + (a_{12} - 2\zeta_2 u^2)ug' = 0,$$

$$((1 - 4\gamma)a_{11} + (7 - 4\gamma)\xi_1 - 2a_{22})g + (a_{11} - 2a_{22} - \xi_1)ug' = 0.$$

Dividing the fourth equation by g (as it is nonzero) and differentiating it with respect to u , one is left to study the following cases:

(a) $\left(\frac{ug'}{g}\right)' \neq 0$ and

(b) $\left(\frac{ug'}{g}\right)' = 0$.

Consider $\left(\frac{ug'}{g}\right)' \neq 0$. It follows that $a_{11} = 2a_{22} + \xi_1$. If $\gamma = 0$

then $\xi_1 = 0$, but if $\gamma \neq 0$ then $a_{22} = \xi_1 \left(\frac{1 - \gamma}{\gamma}\right)$. From the third equation, one needs to study the following cases:

- (i) the case where there exists a generator with $a_{12} \neq 0$, and
- (ii) the case for which all generators have $a_{12} = 0$.

If there exists a generator with $a_{12} \neq 0$, then g satisfies the form $(1 - 4\gamma)g + (1 - \beta u^2)ug' = 0$. Notice that $\beta = 0$

is reducible. Hence, $\beta \neq 0$. Without loss of generality, one can assume that $\beta = 1$. Then the general solution of g is $g_0 \left(1 - \frac{1}{u^2}\right)^{\tilde{\gamma}}$ where $\tilde{\gamma} = \frac{1-4\gamma}{2} \neq 0$ (if $\tilde{\gamma} = 0$, the case is reducible). Substituting this into the determining equations, we obtain $\zeta_2 = \frac{a_{12}}{2}$. It follows that $f = \phi(u) \left(1 - \frac{1}{u^2}\right)^{\tilde{\gamma}+(1/2)}$, where $\phi = f_0 - 2g_0 \left(\frac{1}{(u^2-1)^{(1/2)}}\right)$. Here, the extension $X_4 + 2X_7$ is obtained besides the studied subalgebra.

For the case where all generators have $a_{12} = 0$, it follows that $\zeta_2 = 0$. No other extensions are obtained for this case.

Consider $\left(\frac{ug'}{g}\right)' = 0$. The general solution for this is $g = g_0 u^\kappa$.

Substituting this into the determining equations, one obtains that $a_{12} = 0$ and $\zeta_2 = 0$. Consequently, the form of f either satisfies $(\kappa + 1)f - uf' = 0$ or not. If it is satisfied, then the general solution is $f = f_0 u^{\kappa+1}$. Moreover, $a_{22} = (\kappa - 4\gamma + 1)(a_{11} - \xi_1) + 8\xi_1$. Here, the extension $(\kappa + 1)X_2 + 2X_6$ is obtained apart from the studied subalgebra. If f does not satisfy $(\kappa + 1)f - f'u = 0$, then no extensions are obtained other than the studied subalgebra.

- (3) For the case when $\alpha = 1$, the determining equations after splitting with respect to y are as follows

$$\begin{aligned} (1 - 2\gamma)\zeta_1 f + (\zeta_1 - \zeta_2 u)uf' &= 0, \\ ((1 - 2\gamma)a_{12} + ((4 - 2\gamma)\xi_1 u - 2\gamma a_{11} u))f - a_{12}ug \\ + ((a_{11} - a_{22})u + a_{12} - a_{21}u^2)uf' &= 0, \\ (1 - 2\gamma)\zeta_1 g + (\zeta_1 - \zeta_2 u)ug' &= 0, \\ -a_{21}uf + g((1 - 2\gamma)a_{11}u + (1 - 2\gamma)a_{12} + (4 - 2\gamma)\xi_1 u - a_{22}u) \\ + g'u((a_{11} - a_{22})u + a_{12} - a_{21}u^2) &= 0. \end{aligned}$$

From the first and third equations, one can study the following 2 cases:

- (a) $fg' - gf' = 0$, and
- (b) $fg' - gf' \neq 0$.

Notice that when $fg' - gf' = 0$, then $g = g_0 f$. This is a reducible

case. Hence, we consider only when $fg' - gf' \neq 0$. It follows that $\zeta_1 = \zeta_2 = 0$. From here, one can assume that $g = \phi(u)f$ (as f is nonzero), where $\phi' \neq 0$. If it is assumed further that $\phi = \psi(u) + 1/u$, then the determining equations are reduced as follows:

$$\begin{aligned} & (2(-\gamma a_{11}u + (2 - \gamma)\xi_1u) - (\gamma + \psi u)a_{12})f \\ & + ((a_{11} - a_{22})u + a_{12} - a_{21}u^2)uf' = 0, \\ & (a_{11} - a_{22})u + a_{12} - a_{21}u^2)\psi' + a_{12}\psi^2 + (a_{11} + 2a_{12}u^{-1} - a_{22})\psi = 0. \end{aligned} \quad (25)$$

These equations lead one to study the two cases where:

- (i) there exists at least one generator with $a_{12} \neq 0$, and
- (ii) where all generators have $a_{12} = 0$.

For the case where there exists at least one generator with $a_{12} \neq 0$, it follows that $\psi(u) = -\frac{\kappa u^2 + \lambda u + \beta}{u(\beta - \psi_0 u)}$, where $\beta \neq 0$, $\psi_0 \neq 0$, λ, κ are constant. Without loss of generality, it is assumed further that $\beta = 1$. Consequently, we obtain $a_{11} = \lambda a_{12} + a_{22}$ and $a_{21} = -\kappa a_{12}$. Substituting this into the remaining determining equations gives the solution for f which depends on the following three cases:

- (A) $4\kappa - \lambda^2 > 0$,
- (B) $4\kappa - \lambda^2 < 0$, and
- (C) $4\kappa - \lambda^2 = 0$.

For the case where $4\kappa - \lambda^2 > 0$, it is assumed that $4\kappa - \lambda^2 = p^2$, $p \neq 0$. The solution for f is

$$f_0 \frac{(1 - \psi_0 u)u^{2\gamma-1}}{(\kappa u^2 + \lambda u + 1)^\gamma} e^{\left(\frac{(2\lambda\gamma - 4\mu)}{p} \arctan\left(\frac{\lambda + 2\kappa u}{p} \right) \right)}$$

where μ is constant.

For the case where $4\kappa - \lambda^2 < 0$, it is assumed that $4\kappa - \lambda^2 = -p^2$, $p \neq 0$. The solution for f is

$$f_0 \frac{(1 - \psi_0 u)u^{2\gamma-1}}{(\kappa u^2 + \lambda u + 1)^\gamma} \left(\frac{2\kappa u + \lambda - p}{2\beta\kappa u + \lambda + p} \right)^{\frac{\lambda\gamma - 2\mu}{p}}$$

where μ_0 is constant.

For the case where $4\kappa - \lambda^2 = 0$, it follows that

$$f = f_0 \frac{(1 - \psi_0 u) u^{2\gamma-1}}{(\kappa u^2 + \lambda u + 1)^\gamma} e^{\left(-\frac{4(\gamma + \mu u)}{\lambda u + 2}\right)}$$

where μ is constant.

If $\gamma \neq 0$, then $a_{22} = \frac{(2 - \gamma)\xi_1 - \mu a_{12}}{\gamma}$. If $\gamma = 0$, then $\xi_1 = \frac{\mu a_{12}}{2}$. Here, the extension $(\lambda\gamma - \mu)X_5 - \mu X_6 + \gamma X_7 - \kappa\gamma X_8$ is obtained apart from the studied subalgebra.

For the case where all generators have $a_{12} = 0$, the determining equations are reduced to

$$\begin{aligned} 2((2 - \gamma)\xi_1 - \gamma a_{11})f + (a_{11} - a_{22} - a_{21}u)uf' &= 0, \\ ((4 - 2\gamma)\xi_1 - a_{22} + (1 - 2\gamma)a_{11})g - a_{21}f & \\ + (a_{11} - a_{22} - a_{21}u)ug' &= 0. \end{aligned} \quad (26)$$

Dividing the first equation (26) by uf' and differentiating this equation with respect to u twice, leads to the study of the following sub-cases:

(A) $\left(\frac{f}{uf'}\right)'' \neq 0$, and

(B) $\left(\frac{f}{uf'}\right)'' = 0$.

If $\left(\frac{f}{uf'}\right)'' \neq 0$, then it follows that if $\gamma \neq 0$ then $a_{11} = \xi_1 \frac{2 - \gamma}{\gamma}$, $a_{22} = \xi_1 \frac{2 - \gamma}{\gamma}$ and $a_{21} = 0$. If $\gamma = 0$ then $\xi_1 = 0$, $a_{22} = a_{11}$ and $a_{21} = 0$. For both cases, no extensions are obtained apart from the studied subalgebra.

If $\left(\frac{f}{f'u}\right)'' = 0$, then the general solution for f is $f_0 \left(\frac{u}{1 + u}\right)^\kappa$, where $\kappa \neq 0$ (else it is reducible). Substituting this into the determining equations (26), one obtains that $a_{21} = 2 \left(\frac{-\gamma a_{11} + (2 - \gamma)\xi_1}{\kappa}\right)$ and $a_{22} = \frac{(\kappa - 2\gamma)a_{11} + (4 - 2\gamma)\xi_1}{\kappa}$. Substituting this into the remaining determining equation,

one finds that g satisfies $g'u(1+u) + (1-\kappa)g + f = 0$. The general solution of this is $g = \left(g_0 - f_0 \frac{u}{u+1}\right) \left(\frac{u}{u+1}\right)^{\kappa-1}$. The extension $\kappa X_2 + 2(X_6 + X_8)$ is obtained aside from the studied subalgebra.

- (4) For the case where $\alpha \neq 0, \frac{1}{2}, 1$, the determining equations are split with respect to y . Since $f' \neq 0$, it follows that $\zeta_2 = 0$ and $a_{21} = 0$. Notice also that since $\alpha u g' + (\alpha - 2\gamma)g = 0$ leads to a degenerate case, then $\zeta_1 = 0$ and $a_{12} = 0$. The remaining determining equations become

$$\begin{aligned} (\alpha a_{11} + (\alpha - 1)\xi_1 - a_{22})uf' + (-2\gamma a_{11} + (4 - 2\gamma)\xi_1)f &= 0, \\ (\alpha a_{11} + (\alpha - 1)\xi_1 - a_{22})ug' + ((\alpha - 2\gamma)a_{11} + (\alpha - 2\gamma + 3)\xi_1 - a_{22})g &= 0. \end{aligned} \quad (27)$$

Dividing the first equation by f (as it is nonzero) and differentiating with respect to u , it can be observed that there is need to study the following cases:

- (a) $\left(\frac{uf'}{f}\right)' \neq 0$ and
(b) $\left(\frac{uf'}{f}\right)' = 0$.

For the case with $\left(\frac{uf'}{f}\right)' \neq 0$, it follows that $a_{22} = \alpha a_{11} + (\alpha - 1)\xi_1$. Substituting this into the remaining determining equations (27), we find that if $\gamma \neq 0$ then $a_{11} = \frac{2-\gamma}{\gamma}\xi_1$, and if $\gamma = 0$ then $\xi_1 = 0$. Substituting all these, no other extensions of the generator is found other than the studied subalgebra.

For the case where $\left(\frac{uf'}{f}\right)' = 0$, the general solution for f is $f_0 u^\kappa$, where $\kappa \neq 0$. Substituting this function into the determining equations, it follows that $a_{22} = \frac{(\kappa\alpha - 2\gamma)a_{11} + (\kappa\alpha - \kappa - 2\gamma + 4)\xi_1}{\kappa}$. This leads us to study the two cases, that is, if $g'u - g(\kappa - 1) = 0$ or $g'u - g(\kappa - 1) \neq 0$. If $g'u - g(\kappa - 1) = 0$, the general solution for g is $g_0 u^{\kappa-1}$. Another extension of the generator apart from the studied subalgebra is found, that is $\kappa X_2 + 2X_6$. For the case where

$g'u - g(\kappa - 1) \neq 0$, if $\gamma \neq 0$ then $a_{11} = \frac{2-\gamma}{\gamma}\xi_1$, and if $\gamma = 0$ then $\xi_1 = 0$. Again, these lead to a generator with only the studied subalgebra as its extension.

The complete representative classes for the autonomous system with all admitted generators having $\xi'' = 0$ is listed in Tables 2 and 3.

4 Conclusion

A complete group classification of the systems of two autonomous nonlinear second-order ordinary differential equations of the form $\mathbf{y}'' = \mathbf{F}(\mathbf{y})$ excluding the systems which are equivalent to linear systems and the degenerate case were presented using both the direct and algebraic approach. The important thing in this study is that the analysis of the determining equations were split into two cases: 1) the case where at least one admitted generator has $\xi'' \neq 0$ and 2) the case where all admitted generators have $\xi'' = 0$. The first was analyzed through the direct approach while the latter was analyzed using one-dimensional optimal system of subalgebras followed by the direct approach. For the direct approach, all possible Lie algebras were found with the aid of the equivalence transformations applied on the determining equations. As for the algebraic approach, the study was reduced to the analysis of relations between constants of the generator with its corresponding basis operators. The obtained classification is summarized on Tables 1, 2 and 3. It is highly likely that the same methods shown in this paper are applicable to the group classification of systems of two nonlinear second-order ordinary differential equations, which will be next goal for further studies. In addition, it is also believed that this can be extended to systems in more general cases.

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Table 1

Group classification of systems admitting at least one generator with $\xi'' \neq 0$.

F	G	κ	Extension of Kernel
$\kappa y + \frac{f_0 y}{z^4} \left(\frac{z}{y}\right)^{-4/(\gamma-1)}$	$\kappa z + \frac{f_1}{z^3} \left(\frac{z}{y}\right)^{-4/(\gamma-1)}$	0	Y_2, Y_3, Y_4
		-1	Y_7, Y_8, Y_4
		1	Y_9, Y_{10}, Y_4
$\kappa y + (f_0 y - f_1 z)\tau(y, z)$	$\kappa z + (f_0 z + f_1 y)\tau(y, z)$	0	Y_2, Y_3, Y_5
		-1	Y_7, Y_8, Y_5
		1	Y_9, Y_{10}, Y_5
$\kappa y + e^{\frac{y}{z}} z^{-4} (f_0 y + f_1 z)$	$\kappa z + f_0 z^{-3} e^{\frac{y}{z}}$	0	Y_2, Y_3, Y_6
		-1	Y_7, Y_8, Y_6
		1	Y_9, Y_{10}, Y_6

Table 2

Group classification of systems admitting all generator with $\xi'' = 0$. Here we have $\theta_1(u, v) = (\cos(u)f(v) + \sin(u)g(v))$, $\theta_2(u, v) = \sin(u)f(v) - \cos(u)g(v)$, $\chi_1(\alpha) = \frac{\alpha}{\alpha^2 + 1}$ and $\chi_2(\alpha) = \frac{1}{\alpha^2 + 1}$.

F	G	Relations	Extension of Kernel
$f(u)y^{(1-2\gamma)}$	$g(u)y^{(\alpha-2\gamma)}$	$u = \frac{y^\alpha}{z} \quad -1 \leq \alpha \leq 1$	$\gamma X_2 + X_5 + \alpha X_6$
$f(u)y^{(1-2\gamma)}$	$g(u)y^{(-2\gamma)}$	$u = ye^{-z}$	$\gamma X_2 + X_4 + X_5$
$e^{-2\gamma u}\theta_1(u, v)$	$-e^{-2\gamma u}\theta_2(u, v)$	$y = v \cos(u), \quad z = v \sin(u)$	$\gamma X_2 - X_7 + X_8$
$e^{(\alpha-2\gamma)u}\theta_1(u, v)$	$e^{(\alpha-2\gamma)u}\theta_2(u, v)$	$y = ve^{\alpha u} \cos(u) + \chi_1(\alpha)$ $z = ve^{\alpha u} \sin(u) - \chi_2(\alpha), \quad \alpha > 0$	$\gamma X_2 - X_3 + \alpha(X_5 + X_6) - X_7 + X_8$
$e^{(\alpha-2\gamma)u}\theta_1(u, v)$	$e^{(\alpha-2\gamma)u}\theta_2(u, v)$	$y = ve^{\alpha u} \cos(u) - \chi_1(\alpha)$ $z = ve^{\alpha u} \sin(u) + \chi_2(\alpha), \quad \alpha > 0$	$\gamma X_2 + X_3 + \alpha(X_5 + X_6) - X_7 + X_8$
$e^{(\alpha-2\gamma)u}\theta_1(u, v)$	$e^{(\alpha-2\gamma)u}\theta_2(u, v)$	$y = ve^{\alpha u} \cos(u)$ $z = ve^{\alpha u} \sin(u), \quad \alpha > 0$	$\gamma X_2 + \alpha(X_5 + X_6) - X_7 + X_8$
$(g(v)u + f(v))e^{(-2\gamma u)}$	$g(v)e^{(-2\gamma u)}$	$y = uv, \quad z = v$	$\gamma X_2 + X_7$
$(g(u)z + f(u))e^{(-2\gamma z)}$	$g(u)e^{(-2\gamma z)}$	$u = z^2 - 2y$	$\gamma X_2 + X_4 + X_7$
$((y/z)g(u) + f(u))e^{((1-2\gamma)(y/z))}$	$g(u)e^{((1-2\gamma)(y/z))}$	$u = ze^{-y/z}$	$\gamma X_2 + X_5 + X_6 + X_7$
$f(z)e^{-2\gamma y}$	$g(z)e^{-2\gamma y}$		$\gamma X_2 + X_3$

Table 3: Group classification of systems admitting all generator with $\xi'' = 0$. Here, $f_0, g_0, \phi_0, \phi_1, \alpha, \beta, \kappa, \mu_0, \lambda$ and γ are constant.

Subalgebra 1. $\gamma X_2 + X_5$			
F	G	Relations	Additional Extension of Kernel
$f_0 z^\beta y^{1+\tilde{\gamma}}$	$g_0 z^{\beta+1} y^{\tilde{\gamma}}$	$\tilde{\gamma} = -2\gamma \neq 0, \beta \neq 0$	$\beta X_5 - \tilde{\gamma} X_6$
$f_0 y^{1+\tilde{\gamma}} e^{\kappa z}$	$g_0 y^{\tilde{\gamma}} e^{\kappa z}$	$\tilde{\gamma} = -2\gamma \neq 0, \kappa \neq 0$	$\kappa X_5 - \tilde{\gamma} X_4$
Subalgebra 1. $\gamma X_2 + X_5 + \frac{1}{2} X_6$			
F	G	Relations	Additional Extension of Kernel
$\phi(u)(y - z^2)^{\tilde{\gamma}}$	$g_0(y - z^2)^{\tilde{\gamma}}$	$\tilde{\gamma} = \frac{1-4\gamma}{2} \neq 0$	
		$\phi = f_0(y - z^2)^{1/2} + 2g_0 z$	$X_4 + 2X_7$
$f_0 z^{-(\kappa+1)} y^{\tilde{\gamma}+1}$	$g_0 z^{-\kappa} y^{\tilde{\gamma}}$	$\kappa + 1 \neq 0, \tilde{\gamma} = \frac{\kappa+1-4\gamma}{2} \neq 0$	$(\kappa+1)X_2 + 2X_6$
Subalgebra 1. $\gamma X_2 + X_5 + X_6$			
F	G	Relations	Additional Extension of Kernel
$f_0 \frac{z - \alpha y}{(z^2 + \lambda y z + \kappa y^2)^\gamma} \psi_i(y, z)$	$-f_0 \frac{(\kappa y + (\lambda + \alpha)z)}{(z^2 + \lambda y z + \kappa y^2)^\gamma} \psi_i(y, z)$	$i = 1, 2, 3, \alpha \neq 0,$	$(\lambda\gamma - \mu)X_5 - \mu X_6 + \gamma X_7 - \kappa\gamma X_8$
	Here, $\psi_1(y, z) = e^{\frac{2\lambda\gamma - 4\mu}{p} \arctan \frac{\lambda z + 2\kappa y}{pz}}$ with $4\kappa - \lambda^2 = p^2, p \neq 0;$		
	$\psi_2(y, z) = \left(\frac{2\kappa y + (\lambda + p)z}{2\kappa y + (\lambda - p)z} \right)^{\frac{2\mu - \lambda\gamma}{p}}$ with $4\kappa - \lambda^2 = -p^2, p \neq 0;$ and		
	$\psi_3(y, z) = e^{-\frac{4(\mu y + \gamma z)}{\lambda y + 2z}}$ with $4\kappa - \lambda^2 = 0$		
$f_0 \left(\frac{y}{y+z} \right)^\kappa y^{1-2\gamma}$	$\left(g_0 - f_0 \frac{y}{y+z} \right) \left(\frac{y}{y+z} \right)^{\kappa-1} y^{1-2\gamma}$	$\gamma \neq 0, \kappa \neq 0$	$\kappa X_2 + 2(X_6 + X_8)$
Subalgebra 1. $\gamma X_2 + X_5 + \alpha X_6, \alpha \neq 0$			
F	G	Relations	Additional Extension of Kernel
$f_0 z^{-\kappa} y^{\tilde{\gamma}+1}$	$g_0 z^{1-\kappa} y^{\tilde{\gamma}}$	$\tilde{\gamma} = \alpha\kappa - 2\gamma, \alpha \neq 0, 1/2, 1, \kappa \neq 0$	$\kappa X_2 + 2X_6$
Table 3 – continued on next page			

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Subalgebra 2. $\gamma X_2 + X_4 + X_5$			
F	G	Relations	Additional Extension of Kernel
$f_0 y^{(\kappa+1)} e^{-\alpha z}$	$g_0 y^{(\kappa)} e^{-\alpha z}$	$\gamma = \frac{\alpha - \kappa}{2}, \kappa \alpha \neq 0$	$\alpha X_5 + \kappa X_4$
Subalgebra 3. $-X_7 + X_8$			
F	G	Relations	Additional Extension of Kernel
$(f_0 \cos(u) + g_0 \sin(u)) v^\kappa$	$(f_0 \sin(u) - g_0 \cos(u)) v^\kappa$	$f_0 \neq 0, g_0 \neq 0,$ $u = \arctan(z/y), v^2 = y^2 + z^2$	$\frac{1 - \kappa}{2} X_2 + X_5 + X_6$
Subalgebra 3. $\gamma X_2 - X_7 + X_8, \gamma \neq 0$			
F	G	Relations	Additional Extension of Kernel
$e^{\tilde{\gamma} u} (f_0 \cos(u) + g_0 \sin(u)) v^{-\tilde{\gamma} \kappa - 3}$	$e^{\tilde{\gamma} u} (f_0 \sin(u) - g_0 \cos(u)) v^{-\tilde{\gamma} \kappa - 3}$	$f_0 \neq 0, g_0 \neq 0,$ $\tilde{\gamma} = -2\gamma \neq 0, u = \arctan(z/y), v^2 = y^2 + z^2$	$2X_2 + X_5 + X_6 + \kappa(X_8 - X_7)$
Subalgebra 4. $\gamma X_2 - X_3 + \alpha(X_5 + X_6) - X_7 + X_8, \alpha > 0$			
F	G	Relations	Additional Extension of Kernel
$e^{(\alpha - 2\gamma)u} (f_0 \cos(u) + g_0 \sin(u)) v^\kappa$	$e^{(\alpha - 2\gamma)u} (f_0 \sin(u) - g_0 \cos(u)) v^\kappa$	$f_0 \neq 0, g_0 \neq 0,$ $u = \arctan\left(\frac{z + \chi_2(\alpha)}{y - \chi_1(\alpha)}\right),$ $v^2 = e^{-2\alpha u} ((y - \chi_1(\alpha))^2 + (z + \chi_2(\alpha))^2),$ $\chi_1(\alpha) = \frac{\alpha}{\alpha^2 + 1}, \chi_2(\alpha) = \frac{1}{\alpha^2 + 1}$	$\frac{1 - \kappa}{2} X_2 + X_5 + X_6 - \chi_1 X_3 + \chi_2 X_4$
Subalgebra 4. $\gamma X_2 + X_3 + \alpha(X_5 + X_6) - X_7 + X_8, \alpha > 0$			
Table 3 – continued on next page			

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F	G	Relations	Additional Extension of Kernel
$e^{(\alpha-2\gamma)u}(f_0 \cos(u) + g_0 \sin(u))v^\kappa$	$e^{(\alpha-2\gamma)u}(f_0 \sin(u) - g_0 \cos(u))v^\kappa$	$f_0 \neq 0, g_0 \neq 0,$ $u = \arctan\left(\frac{z - \chi_2(\alpha)}{y + \chi_1(\alpha)}\right),$ $v^2 = e^{-2\alpha u}((y + \chi_1(\alpha))^2 + (z - \chi_2(\alpha))^2),$ $\chi_1(\alpha) = \frac{\alpha}{\alpha^2 + 1}, \chi_2(\alpha) = \frac{1}{\alpha^2 + 1}$	$\frac{1 - \kappa}{2}X_2 + X_5 + X_6 + \chi_1 X_3 - \chi_2 X_4$
Subalgebra 4. $\gamma X_2 + \alpha(X_5 + X_6) - X_7 + X_8, \alpha > 0$			
F	G	Relations	Additional Extension of Kernel
$e^{(\alpha-2\gamma)u}(f_0 \cos(u) + g_0 \sin(u))v^\kappa$	$e^{(\alpha-2\gamma)u}(f_0 \sin(u) - g_0 \cos(u))v^\kappa$	$f_0 \neq 0, g_0 \neq 0,$ $u = \arctan(z/y), v^2 = e^{-2\alpha u}(y^2 + z^2)$	$\frac{1 - \kappa}{2}X_2 + X_5 + X_6$
Subalgebra 5. $\gamma X_2 + X_7$			
F	G	Relations	Additional Extension of Kernel
$g_0 z^{\beta-1} e^{-y/z}(y + \kappa \tilde{\gamma} z)$	$g_0 z^\beta e^{-y/z}$	$\tilde{\gamma} = 2\gamma$	$X_5 + \tilde{\gamma} X_6 + (\beta - 1)X_7$
Subalgebra 5. $\gamma X_2 + X_4 + X_7$			
F	G	Relations	Additional Extension of Kernel
$(g_0 z + f_0)e^{(\beta u - 2\gamma z)}$	$g_0 e^{(\beta u - 2\gamma z)}$	$u = z^2 - 2y, \beta \neq 0$	$\beta X_2 + X_3$
Subalgebra 5. $X_4 + X_7$			
F	G	Relations	Additional Extension of Kernel
$(g_0 z + f_0(\beta + u)^{1/2})(\beta + u)^\kappa$	$g_0(\beta + u)^\kappa$	$u = z^2 - 2y, \kappa \neq 0$	$(1 - 2\kappa)X_2 + 2(-\beta X_3 + 2X_5 + X_6)$
Subalgebra 6. $\gamma X_2 + X_5 + X_6 + X_7$			
Table 3 – continued on next page			

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F	G	Relations	Additional Extension of Kernel
$(g_0y + f_0z)z^{\kappa-1}e^{-\tilde{\gamma}(y/z)}$	$g_0z^{\kappa}e^{-\tilde{\gamma}(y/z)}$	$\tilde{\gamma} = 2\gamma + \kappa - 1 \neq 0$	$(\kappa - 1)X_2 - 2(X_5 + X_6)$
Subalgebra 7. $\gamma X_2 + X_3, \gamma \neq 0$			
F	G	Relations	Additional Extension of Kernel
$(f_0z^{\beta-1}e^{\kappa z - \tilde{\gamma}y})(\kappa z + \tilde{\gamma}\phi_1)$	$g_0z^{\beta}e^{\kappa z - \tilde{\gamma}y}$	$\tilde{\gamma} = 2\gamma, f_0 = g_0/\tilde{\gamma}$	$(\beta - 1)X_3 + \kappa X_7 + \tilde{\gamma}X_6$
$(g_0e^{\beta z + \kappa z^2 - \tilde{\gamma}y})\phi(z)$	$g_0e^{\beta z + \kappa z^2 - \tilde{\gamma}y}$	$\tilde{\gamma} = 2\gamma,$	
		$\phi = \phi_0z + \phi_1, \phi_0 \neq 0, \kappa = (\tilde{\gamma}\phi_0)/2$	$\beta X_3 + 2\kappa X_7 + \tilde{\gamma}X_4$

Appendix 1. For $\xi'' = 0$, the determining equations have the form

$$\begin{cases} \zeta_1'' = F_y \zeta_1 + F_z \zeta_2 + q_1, \\ \zeta_2'' = G_y \zeta_1 + G_z \zeta_2 + q_2, \end{cases}$$

where q_1 and q_2 are functions of y and z . Differentiating them with respect to x , one has

$$\begin{cases} \zeta_1''' = F_y \zeta_1' + F_z \zeta_2', \\ \zeta_2''' = G_y \zeta_1' + G_z \zeta_2', \end{cases}$$

Differentiating the latter equations with respect to y and z

$$\begin{cases} F_{yy} \zeta_1' + F_{yz} \zeta_2' = 0, \\ F_{yz} \zeta_1' + F_{zz} \zeta_2' = 0, \\ G_{yy} \zeta_1' + G_{yz} \zeta_2' = 0, \\ G_{yz} \zeta_1' + G_{zz} \zeta_2' = 0, \end{cases}$$

Case 1. Let $F_{zz} \neq 0$, then

$$\zeta_2' = -\frac{F_{yz}}{F_{zz}} \zeta_1', \quad F_{yy} - \frac{F_{yz}^2}{F_{zz}} = 0, \quad G_{yy} - G_{yz} \frac{F_{yz}}{F_{zz}} = 0, \quad G_{yz} - G_{zz} \frac{F_{yz}}{F_{zz}} = 0.$$

Thus,

$$\frac{F_{yz}}{F_{zz}} = k,$$

and

$$F_{yy} - kF_{yz} = 0, \quad G_{yy} - kG_{yz} = 0, \quad G_{yz} - kG_{zz} = 0$$

or

$$(F_y - kF_z)_y = 0, \quad (F_y - kF_z)_z = 0, \quad (G_y - kG_z)_y = 0, \quad (G_y - kG_z)_z = 0,$$

One has

$$\begin{aligned} F_y - kF_z &= k_1, \quad G_y - kG_z = k_2, \\ \frac{dy}{1} &= \frac{dz}{-k} = \frac{dF}{k_1}, \quad F = \Phi(z + ky) + k_1 y, \\ \frac{dy}{1} &= \frac{dz}{-k} = \frac{dG}{k_2}, \quad G = \Psi(z + ky) + k_2 y. \end{aligned}$$

Changing the variables

$$\bar{y} = y, \quad \bar{z} = z + ky,$$

the original system

$$y'' = F(y, z), \quad z'' = G(y, z),$$

becomes

$$y'' = \Phi(\bar{z}) + k_1 y, \quad \bar{z}'' = (k\Phi(\bar{z}) + \Psi(\bar{z})) + (kk_1 + k_2)y$$

Thus one needs to study the equations

$$y'' = k_1 y + F(z), \quad z'' = k_2 y + G(z), \quad k_2 F'' \neq 0.$$

$$\begin{cases} \zeta_1'' = k_1 \zeta_1 + F' \zeta_2 + q_1, \\ \zeta_2'' = k_2 \zeta_1 + G' \zeta_2 + q_2, \end{cases}$$

Notice that because of $F'' \neq 0$, then $\zeta_2' = 0$ from the second equation

$$0 = k_2 \zeta_1' \Rightarrow \zeta_1' = 0.$$

Case 2. Let $F_{zz} = 0$, then by symmetry $G_{yy} = 0$. Hence

$$F_{yy}\zeta_1' + F_{yz}\zeta_2' = 0, \quad F_{yz}\zeta_1' = 0, \quad G_{yz}\zeta_2' = 0, \quad G_{yz}\zeta_1' + G_{zz}\zeta_2' = 0,$$

If

$$F_{yz} \neq 0 \Rightarrow \zeta_1' = 0, \quad \zeta_2' = 0.$$

Hence,

$$F_{yz} = 0, \quad F_{zz} = 0, \quad G_{yy} = 0, \quad G_{yz} = 0$$

and

$$F_{yy}\zeta_1' = 0, \quad G_{zz}\zeta_2' = 0.$$

Thus

$$y'' = k_1 z + F(y), \quad z'' = k_2 y + G(z), \quad k_1 k_2 (F''^2 + G''^2) \neq 0.$$

$$\begin{cases} \zeta_1'' = F' \zeta_1 + k_1 \zeta_2 + q_1, \\ \zeta_2'' = k_2 \zeta_1 + G' \zeta_2 + q_2, \end{cases}$$

Let $F'' \neq 0$, then $\zeta'_1 = 0$ and because

$$\zeta'''_1 = F'\zeta'_1 + k_1\zeta'_2 \Rightarrow k_1\zeta'_2 = 0 \Rightarrow \zeta'_2 = 0$$

Let $G'' \neq 0$, then $\zeta'_2 = 0$ and because

$$\zeta'''_2 = k_2\zeta'_1 + G'\zeta'_2 \Rightarrow k_2\zeta'_1 = 0 \Rightarrow \zeta'_1 = 0$$